Lindelöf spaces $C(X)$ over topological groups

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(Communicated by Karl-Hermann Neeb)

Abstract. Theorem 1 proves (among the others) that for a locally compact topological group X the following assertions are equivalent: (i) X is metrizable and σ-compact. (ii) $C_p(X)$ is analytic. (iii) $C_p(X)$ is K-analytic. (iv) $C_p(X)$ is Lindelöf. (v) $C_c(X)$ is a separable metrizable and complete locally convex space. (vi) $C_c(X)$ is *compactly dominated by irrationals*. This result supplements earlier results of Corson, Christensen and Calbrix and provides several applications, for example, it easily applies to show that: (1) For a compact topological group X the Eberlein, Talagrand, Gul'ko and Corson compactness are equivalent and any compact group of this type is metrizable. (2) For a locally compact topological group X the space $C_p(X)$ is Lindelöf iff $C_c(X)$ is weakly Lindelöf. The proofs heavily depend on the following result of independent interest: A locally compact topological group X is metrizable iff every compact subgroup of X has countable tightness (Theorem 2). More applications of Theorem 1 and Theorem 2 are provided.

1991 Mathematics Subject Classification: 22A05, 43A40, 54H11.

1 Introduction

By $C_p(X)$ and $C_c(X)$ we denote the space of continuous real-valued maps on a Tychonoff space X endowed with the pointwise and compact-open topology, respectively.

One of the unsolved problems in the theory of spaces $C_p(X)$ asks ([5, Problem 44, p. 29]) *when exactly for a given X the space $C_p(X)$ is Lindelöf*. It is well known that, for example, if X is second countable, then $C_p(X)$ is Lindelöf [26, (3.8.D)]. The same conclusion holds also for (not necessarily second countable) Corson compact spaces

The research for the first named author was supported by the Komitet Badań Naukowych (State Committee for Scientific Research), Poland, grant no. 2P03A 022 25 and by the grant of the Spanish Ministry of Education and Science SAB2004-0025 to support his stay at the Technical University of Valencia (Spain). This research was also supported by the Spanish Ministry of Education, projects MTM 2005-01182 for the first two named authors and BFM 2003-05878 for the last two named authors. These are FEDER projects, cofinanced by the European Community.

X (Alster-Pol-Gul'ko's theorem [4, IV.2.22]). We refer the reader to [4, 5, 7, 13, 14, 24] for some other known (positive and negative) results about this problem. Theorem 1 below provides several equivalent conditions for $C_p(X)$ to be Lindelöf when X is a *locally compact group*. Before its formulation we recall that $C_p(X)$ and $C_c(X)$ are locally convex spaces and that the weak topology of the latter is in general strictly finer than the topology of $C_p(X)$.

Theorem 1. *For a locally compact topological group X the following assertions are equivalent*:

(1) $C_p(X)$ *is analytic.*

(2) $C_p(X)$ *is K-analytic.*

(3) $C_p(X)$ *is Lindelöf.*

(4) X *is metrizable and σ-compact.*

(5) X *is analytic.*

(6) $C_p(X)$ *is boundedly dominated by irrationals and X is metrizable.*

(7) $C_c(X)$ *is a metrizable, complete and separable locally convex space.*

(8) $C_c(X)$ *is compactly dominated by irrationals.*

(9) $C_c(X)$ *is boundedly dominated by irrationals and X is metrizable.*

(10) $C_c(X)$ *is weakly Lindelöf, i.e. Lindelöf for the weak topology of $C_c(X)$.*

This easily implies that for a compact topological group X the Eberlein, Talagrand, Gul'ko and Corson compactness are equivalent and any compact group of this type is metrizable.

A result related with Theorem 1 contained in [19, Theorem 2] states that a locally compact topological group X is metrizable iff the Banach space $C_0(X)$ of continuous, complex valued functions which vanish at infinity is weakly Lindelöf.

It is clear from Theorem 1 that analyticity, K-analyticity and the Lindelöf property for $C_c(X)$ over locally compact topological groups are also equivalent conditions to the previous ones. Moreover, Theorem 1 shows that for a locally compact topological group X the space $C_p(X)$ is analytic (K-analytic) iff $C_c(X)$ is weakly analytic (weakly K-analytic). This provides a variant of Talagrand's Theorem 3.4 of [42] and Canela's Proposition 2.2 of [9] (where the *K-analytic* case has been proved for compact Hausdorff spaces and paracompact locally compact spaces X, respectively).

The implication (1) $\Rightarrow$ (5) of Theorem 1 is covered by a deep result of Calbrix [8, Theorem 2.3.1], stating that if for a Tychonoff space X the space $C_p(X)$ is analytic, then X must be σ-compact and analytic (cf., also [18, Theorem 3.7] for a weaker result). Note that (8) $\Rightarrow$ (2) follows also from a recent result of Tkachuk [44, Theorem 2.8]: For a Tychonoff space X the space $C_p(X)$ is K-analytic iff it is compactly dominated by irrationals.

Theorem 1 completes the whole picture for spaces $C_p(X)$ and $C_c(X)$ over locally compact groups. The proof of Theorem 1 is transparent, short and elementary. The key role in the proof belongs to Theorem 2 below: A locally compact topological group X is metrizable iff every compact subgroup of X has countable tightness. We provide further interesting applications of Theorem 1 and Theorem 2 (see Proposition 2 and Theorem 3).

2 Definitions and notations

By "a space" we mean "a completely regular Hausdorff space". A continuous image of the space $\mathbb{N}^{\mathbb{N}}$, where the space of integers $\mathbb{N}$ is endowed with the discrete topology, is called an *analytic* space. The following order is considered in $\mathbb{N}^{\mathbb{N}} : (n_k) \leq (m_k)$ if $n_k \leq m_k$ for all $k \in \mathbb{N}$. A continuous image of a space of type $K_{\sigma\delta}$ is called a *K-analytic space* [4, p. 7] (see also [40]). A space X is K-analytic iff it is a continuous image of a Lindelöf Čech-complete space. Clearly analytic $\Rightarrow$ K-analytic $\Rightarrow$ Lindelöf. A space X is *compactly dominated by irrationals* if it can be covered by an ordered family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets, i.e. $K_\alpha \subset K_\beta$ if $\alpha \leqq \beta$. Every K-analytic space X is compactly dominated by irrationals (see [43], or [44, Theorem 2.1(g)]), although the converse fails in general [10], see also Example 4 below. A topological group X is called *trans-separable* [25] if X is covered by countably many translations of each neighborhood of the unit of X.

A subset B in a (real or complex) topological vector space E is called *bounded* if B is absorbed by each neighborhood of zero of E. A topological vector space E will be called *boundedly dominated by irrationals* if E is covered by a family $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of bounded sets with the ordering as above.

A space X has *countable tightness* if for every set $A \subset X$ and every $x \in \bar{A}$ there is a countable subset in A whose closure contains x.

A compact space X is said to be: *Eberlein compact* if X is homeomorphic to a weakly compact subset of a Banach space, *Corson compact* if X is homemorphic to a compact subset of a Σ-product of real lines, *Talagrand compact* if $C_p(X)$ is K-analytic and *Gul'ko compact* if $C_p(X)$ is countably determined. We refer the reader to [36] for an internal characterization of Eberlein and Corson compacts, to [27] for a good account of relationships among all these notions and to [4, 26, 28] for unexplained terms.

3 Related results and proofs

We shall need the following result.

Fact 1 ([20]; cf. also [15, Theorem 1 and Remark (ii)]). *Let X be a locally compact topological group. Then there exist a compact subgroup G of X, a number $n \in \mathbb{N} \cup \{0\}$, and a discrete subset $D \subset X$ such that X is homeomorphic to the product $\mathbb{R}^n \times D \times G$.*

Next we give a result which will become an important tool in the sequel.

Theorem 2. *For a locally compact topological group X the following assertions are equivalent*: (1) *X is angelic*. (2) *Every compact subgroup of X has countable tightness.* (3) *X is metrizable*. (4) *X has countable tightness.*

Proof. The only non-trivial implication is (2) $\Rightarrow$ (3).

Case 1. *X is compact.*

First proof. By a result of Kuz'minov [34, Theorem] (see also, [16, Corollary]), every compact Hausdorff group X is *dyadic*, i.e. X is a continuous image of a space of the form $\{0,1\}^{\alpha}$, where α is some cardinal number. It is also known that every dyadic Hausdorff space with countable tightness is metrizable [26], 3.12.12(h), p. 231.

Second proof. Assume that *X is a non-metrizable compact group*. We show that X does not have countable tightness. Since X is a non-metrizable compact group, the weight $w(X)$ of X is uncountable. By [41] (see also, [17, Theorem 3.1]) X contains a subset Y homeomorphic to the Cantor cube $\{0,1\}^{w(X)}$. But $\{0,1\}^{\Gamma}$ (where Γ is an index set with $card(\Gamma) = w(X)$) does not have countable tightness. Indeed, the constant function $f(\gamma) = 1$, for all $\gamma \in \Gamma$ belongs to the closure of the Σ-product of the spaces $D_{\gamma} = \{0,1\}$, $\gamma \in \Gamma$. If $\{0,1\}^{\Gamma}$ had countable tightness, f would have countable support, which clearly provides a contradiction.

Case 2. *Assume that X is a locally compact group*. The proof follows from joint consideration of Fact 1 and Case 1. □

Theorem 2 easily applies to deduce that a locally compact topological group X is metrizable if $C_p(X)$ is separable.

Remarks. (1) Arkhangel'skii [3] asked if every compact homogeneous Hausdorff topological space with countable tightness is first countable. Dow in [23], Theorem 6.3, answered this question positively but under PFA. Eberlein compact spaces provide a large class of spaces with countable tightness, and it is known that homogeneous Eberlein compact spaces are first countable [4, III.3.10], but non-metrizable homogeneous Eberlein compact spaces do exist, see [46]. In [30] Gruenhage proved that every Gul'ko compact space contains a dense G_{δ} subset which is metrizable. This implies that any compact group which is Gul'ko compact must be metrizable, although there are examples of Corson compact spaces without any dense metrizable subspace [45]. On the other hand, since every Corson compact space has countable tightness [32, Lemma 1.6 (ii)], Theorem 2 applies to show that: *In the class of topological groups the Eberlein, Talagrand, Gul'ko, Corson compactness are equivalent properties and each such a compact group is metrizable.*

(2) Theorem 2 extends Theorem 12 of [6], where it was shown (under continuum hypothesis (CH)) that a compact group is metrizable if it is sequentially compact. Later, in [22] it was noted that the metrizability of a compact angelic group can be also proved without (CH). The second proof of Theorem 2 presented above was motivated by [22].

(3) For non locally compact topological groups the implication $(1+4) \Rightarrow (3)$ of Theorem 2 may fail; in fact, let X be an infinite-dimensional reflexive separable real Banach space equipped with the weak topology. Then X is a σ-compact angelic locally convex space ([28, p. 39]) whose compact subsets are metrizable and which has countable tightness ([28, Corollary, p. 38]), but X is not metrizable.

(4) The group structure in the implication (4) $\Rightarrow$ (1) of Theorem 2 is essential: the one-point compactification of the space Ψ of Isbell provides an example of a compact Hausdorff space with countable tightness which is not angelic, see [29, p. 54–55].

(5) For non locally compact topological groups the implication (1) $\Rightarrow$ (4) of Theorem 2 may fail; indeed, the weak topology of any (DF)-space (in the sense of Grothendieck) is angelic, see [11], Theorem 11, but there exist (DF)-spaces whose weak topology does not have countable tightness, [12], p. 514.

We shall need the following applicable interesting facts.

Fact 2. (a) *Every Baire topological vector space which is boundedly dominated by irrationals is metrizable.* (b) *Any metrizable topological vector space is boundedly dominated by irrationals.*

Proof. (a) Assume that E is a Baire topological vector space covered by a family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of bounded sets such that $K_\alpha \subset K_\beta$ if $\alpha \leq \beta$. For a finite sequence of natural numbers $(n_1, \dots, n_k)$ put

$$C_{n_1,n_2,\dots,n_k} := \bigcup \{K_\beta : \beta = (m_l), n_j = m_j, j = 1, \dots, k\}.$$

First step. For a fixed $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, set $W_k := C_{n_1,n_2,\dots,n_k}$, $k \in \mathbb{N}$. For every neighbourhood of zero U in E there exists $k \in \mathbb{N}$ such that $W_k \subset 2^k U$.

Indeed, otherwise there exists a neighbourhood of zero U in E such that for every $k \in \mathbb{N}$ there exists $x_k \in W_k$ such that $2^{-k} x_k \notin U$. Since $x_k \in W_k$ for every $k \in \mathbb{N}$, there exists $\beta_k = (m_n^k)_n \in \mathbb{N}^{\mathbb{N}}$ such that $x_k \in K_{\beta_k}$, $n_j = m_j^k$ for $j = 1, 2, \dots, k$. Set $a_n = \max\{m_n^k : k \in \mathbb{N}\}$ for $n \in \mathbb{N}$ and $\gamma = (a_n)$. As $\gamma \geq \beta_k$ for every $k \in \mathbb{N}$, we have $K_{\beta_k} \subset K_\gamma$, and $x_k \in K_\gamma$ for all $k \in \mathbb{N}$. Since K_γ is bounded, then $2^{-k} x_k \to 0$ in E which provides a contradiction.

Second step. In order to prove that E is metrizable, we will construct recursively a countable basis of neighborhoods of zero in E. Write $E = \bigcup_{n \in \mathbb{N}} C_n$ with C_n as above (with just one subindex). Since E is Baire, there exists $n_1 \in \mathbb{N}$ such that C_{n_1} is of the second category and $\operatorname{Int} \bar{C}_{n_1} \neq \emptyset$. So there exists $x_1 \in \bar{C}_{n_1}$ and a zero neighborhood U_1 such that $x_1 + U_1 \subseteq \bar{C}_{n_1}$.

Now $C_{n_1} = \bigcup_{n \in \mathbb{N}} C_{n_1,n}$ and at least for one natural number n_2 the set C_{n_1,n_2} is of the second category and $\operatorname{Int} \bar{C}_{n_1,n_2} \neq \emptyset$. There exist thus $x_2 \in \bar{C}_{n_1,n_2}$ and a zero neighborhood U_2 so that $x_2 + U_2 \subseteq \bar{C}_{n_1,n_2}$. Proceeding on and on we obtain sequences $(n_k) \in \mathbb{N}^{\mathbb{N}}$, $(x_k)_k$ in E and a sequence $(U_k)_k$ of neighbourhoods of zero in E such that $x_k \in \operatorname{Int} \overline{W_k}$ and $x_k + U_k \subset \overline{W_k}$ for all $k \in \mathbb{N}$.

We claim that $(2^{-k} U_k)_k$ forms a countable basis of neighbourhoods of zero in E. Indeed, take an arbitrary zero neighborhood M in E, and another one V closed and balanced such that $V - V \subseteq M$. By the first step, for V and (n_k) thus fixed, there exists $j \in \mathbb{N}$ such that $W_j \subset 2^j V$. Therefore $x_j + U_j \subseteq \overline{W_j}$, implies $U_j \subseteq \overline{W_j} - \overline{W_j} \subseteq 2^j V - 2^j V \subseteq 2^j M$. Hence $2^{-j} U_j \subseteq M$, and the assertion follows.

(b) If E is a metrizable topological vector space with a countable basis of balanced neighbourhoods of zero $(U_n)_n$, set $K_\alpha := \bigcap_k n_k U_k$ for every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$. It is clear that the family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is as required. □

Fact 3. *If X is a paracompact locally compact space for which $C_p(X)$ is boundedly dominated by irrationals, then X is σ-compact.*

Proof. Since X is a topological direct sum of a family $\{X_i : i \in I\}$ of locally compact σ-compact spaces [26, (5.1.27)], then $C_p(X) = \prod_{i \in I} C_p(X_i)$. But $C_p(X)$ is boundedly dominated by irrationals, so I is countable. Indeed, otherwise $\prod_{i \in I} C_p(X_i)$ would contain a closed subspace of the type $\mathbb{R}^A$ for some uncountable A. But $\mathbb{R}^A$ is Baire and Fact 2(a) applies to deduce that $\mathbb{R}^A$ is metrizable, hence A is countable. Thus X, being a countable topological direct sum of σ-compact spaces, is σ-compact. □

Fact 4. *Let X be a hemicompact space whose compact subsets are metrizable. Then*: (a) *$C_c(X)$ is a metrizable separable locally convex space.* (b) *If moreover X is a k-space, then $C_c(X)$ is a complete metrizable separable locally convex space.*

Proof. (a) is [49, Corollary, p. 271], and (a) together with a well-known assertion on k-spaces gives (b).

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Clearly (**1**) implies (**2**), (**2**) implies (**3**), see [40]. Assume that $C_p(X)$ is Lindelöf. By Asanov theorem, see [4], I.4.1, the space X has countable tightness. By Theorem 2 the space X is metrizable. Therefore X is a metrizable space for which $C_p(X)$ is Lindelöf. Hence, by [4, I.4.7], X is separable. This shows that (**3**) implies (**4**). Clearly (**4**) implies (**5**).

(**5**) implies (**4**): Since X is an analytic Baire topological group, Theorem 5.4 of [18] applies to show that X is metrizable (and Lindelöf).

(**4**) implies (**6**): Since X is locally compact and σ-compact, it is hemicompact and therefore $C_c(X)$ is metrizable. Now Fact 2(b) applies to show that $C_c(X)$ is boundedly dominated by irrationals. Hence $C_p(X)$ is boundedly dominated by irrationals as well.

(**6**) implies (**4**): Since, by assumption, X is a locally compact topological group, it is paracompact. Now X is σ-compact by Fact 3.

(**4**) implies (**7**): Since X is locally compact metrizable and σ-compact, by Fact 4 $C_c(X)$ is a separable, metrizable and complete locally convex space, so (**7**) holds.

Clearly (**7**) implies (**8**). If fact, any separable and complete metric space Y is compactly dominated by irrationals. A direct proof: For a countable and dense sequence $(x_n)_n$ in $C_c(X)$, set $K_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{n_k} B(x_j, k^{-1})$, where $B(x_j, k^{-1})$ is the closed ball in $C_c(X)$ with the center at point x_j and radius k^{-1} for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ and all $j, k \in \mathbb{N}$. Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is as required.

(**8**) implies (**9**): Clearly the first part of (**9**) holds. Moreover, by assumption on $C_c(X)$ the space $C_c(X)$ is trans-separable, [39]. Therefore every compact subset of X

is metrizable (see again [39] remark after Corollary 2), hence by Theorem 2 the space X is metrizable, and the second part of (**9**) holds.

Evidently (**7**) implies (**1**). We show now that (**9**) implies (**4**). Assume (**9**). Since X is paracompact and locally compact, the space $C_c(X)$ is Baire, [37, 10.1.26]. Then by Fact 2(a) the space $C_c(X)$ is metrizable. Hence X is hemicompact, therefore (**4**) holds too.

Finally (**7**) implies (**10**) and (**10**) implies (**3**) are evident. □

It is worth noting that in (**6**) of Theorem 1 the condition "$C_p(X)$ is boundedly dominated by irrationals" cannot be replaced by the following one "$C_p(X)$ is σ-bounded", i.e. $C_p(X)$ is covered by a sequence of bounded subsets.

Example 1. For the additive group $\mathbb{Z}$ of integers with the discrete topology the space $C_p(\mathbb{Z})$ is boundedly dominated by irrationals but $C_p(\mathbb{Z})$ is not σ-bounded.

Indeed, note that $C_p(\mathbb{Z})$, as a metrizable locally convex space, is boundedly dominated by irrationals, see Fact 2. Since $\mathbb{R}^{\mathbb{Z}} = C_p(\mathbb{Z}) = C_c(\mathbb{Z})$, we note that $C_p(\mathbb{Z})$ is a Baire locally convex space. But $C_p(\mathbb{Z})$ is not normable, it cannot be σ-bounded. □

The implication (**1**) $\Rightarrow$ (**4**) of Theorem 1 fails for topological groups which are not locally compact as the following example shows.

Example 2. Let X be the strong dual of $\mathbb{R}^{\mathbb{N}}$. Then X is a non-metrizable Montel (DF)-space and yet $C_p(X)$ is analytic.

Indeed, since $\mathbb{R}^{\mathbb{N}}$ is a Frechet-Montel space, X is hemicompact k-space [49, p. 267] whose compact subsets are metrizable. Hence, by Fact 4, $C_c(X)$ is a complete separable metrizable space. Now $C_p(X)$ is a continuous image of $C_c(X)$, therefore it is analytic. Clearly X is not metrizable, nor locally compact. □

Theorem 1 characterizes those locally compact topological groups X for which $C_c(X)$ is compactly dominated by irrationals. It turns out that the compact domination of locally compact topological groups can be characterized as follows.

Proposition 1. *A locally compact topological group X is compactly dominated by irrationals iff it is Lindelöf.*

Proof. If X is a locally compact topological space which is Lindelöf, then clearly X is σ-compact, so it is compactly dominated by irrationals. Indeed, let $(K_n)_n$ be an increasing sequence of compact subsets of X covering X. Put $K_\alpha := K_{a_1}$ for $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$. If X is a locally compact group compactly dominated by irrationals, then X (according to [39]) is trans-separable. If U is a compact neighbourhood of zero, then countably many translations of U cover X. Hence, X is σ-compact. □

Note that in general Proposition 1 fails for locally compact topological spaces.

Example 3 ([44, Example (3.5)]. There exists a locally compact topological space compactly dominated by irrationals which is not Lindelöf, so neither σ-compact.

4 More about metrizability of locally compact groups

This section deals with other applications of Theorem 2. Let $\mathbb{T}$ be the multiplicative group of complex numbers modulus one endowed with the topology induced from $\mathbb{C}$. For a topological Abelian group X we denote by $X^\wedge$ the set of all continuous group homomorphisms (characters) $\varphi : X \to \mathbb{T}$. The coarsest group topology on X for which all elements of $X^\wedge$ are continuous is called the *Bohr topology* and is denoted by $\sigma(X, X^\wedge)$. The symbol $X_c^\wedge$ stands for $X^\wedge$ endowed with the compact-open topology, see [6] for details.

Proposition 2. *Let X be a separable and metrizable Abelian topological group. Then $X_c^\wedge$ is locally compact iff $X_c^\wedge$ is metrizable.*

Proof. It is straightforward to prove that the evaluation map $e : X \times X_c^\wedge \to \mathbb{T}$ defined by $e(x, \phi) = \phi(x)$, $\phi \in X_c^\wedge$, $x \in X$ is sequentially continuous for any topological abelian group X. If furthermore X and $X_c^\wedge$ are metrizable, then e is continuous. By [35, Proposition 1.2], the group $X_c^\wedge$ is locally compact. For the converse, note that every compact subset of $X_c^\wedge$ is metrizable (since X is separable). Now Theorem 2 applies. □

We provide another application of Theorem 2. It is well known that there are non-metrizable absolutely convex weakly compact sets in Banach spaces over the field $\mathbb{K}$ of real or complex numbers. It turns out that the situation is much better if the valued field $\mathbb{K}$ is different from $\mathbb{R}$ and $\mathbb{C}$ (Theorem 3(b)).

If E denotes a topological vector space over a non-trivially valued field $\mathbb{K}$, and E' the dual space, E is called *dually separating* if E' separates points of E. It is clear that E is dually separating iff the weak topology $\sigma(E, E')$ is Hausdorff.

Recall that a non-trivially valued field $\mathbb{K} := (\mathbb{K}, |\ |)$ is *non-archimedean* if $|t + s| \leq \max\{|t|, |s|\}$ for all $t, s \in \mathbb{K}$, see [47]. A subset B of a vector space E over a non-archimedean non-trivially valued field $\mathbb{K}$ is called *absolutely* $\mathbb{K}$*-convex*, if from $x, y \in B$, $t, s \in \mathbb{K}$, and $|t| \leq 1$, $|s| \leq 1$ it follows that $tx + sy \in B$.

If E is a (Hausdorff) topological vector space over a non-archimedean non-trivially valued complete field $\mathbb{K}$, and E contains a non-zero compact absolutely $\mathbb{K}$-convex set, then $\mathbb{K}$ must be locally compact, see [21].

Now we are ready to apply Fact 1 and Theorem 2 to get the following statement.

Theorem 3. *Let E be a Hausdorff topological vector space over a locally compact non-trivially valued field $\mathbb{K}$. Then*:

(a) *If $\mathbb{K}$ is archimedean, then every locally compact subgroup X of E is metrizable.*

(b) *If $\mathbb{K}$ is non-archimedean and E is a metrizable dually separating space, then every absolutely $\mathbb{K}$-convex locally compact subset X of the topological vector space $(E, \sigma(E, E'))$ is metrizable in $\sigma(E, E')$.*

Proof. (a) As $\mathbb{K}$ is archimedean, from Ostrowski's theorem, [47], Theorem 1.2, it follows that $\mathbb{K}$ is either the field of real or complex numbers. Since X is locally compact it is homeomorphic to the product $\mathbb{R}^n \times D \times G$, where D and G are as in Fact 1. The conclusion follows now taking into account that any compact subgroup in a (real or complex) topological vector space is trivial.

(b) Since $(\mathbb{K}, +)$ is a locally compact Abelian group, $\mathbb{K}^\wedge$ separates the points of $\mathbb{K}$. Fix a non-constant $\chi \in \mathbb{K}^\wedge$. Then $E^\wedge = \{\chi \circ x' : x' \in E'\}$ (cf. [50, Theorem 2]). Since E' separates points of E, we deduce that the Bohr topology of the group $(E, +)$ is Hausdorff and it is coarser than the weak topology $\sigma(E, E')$. Since E is metrizable, by Theorem 11 of [6] the group $(E, \sigma(E, E^\wedge))$ is angelic and from [28, Theorem p. 31] the space $(E, \sigma(E, E'))$ is also angelic. On the other hand, since $\mathbb{K}$ is non-archimedean and X is an absolutely $\mathbb{K}$-convex subset of E, it is an additive subgroup of E. Consequently, X is an angelic locally compact group, and by Theorem 2, we note that X is metrizable. □

Remark. Theorem 3(a) fails for a non-archimedean $\mathbb{K}$. In fact, let $\mathbb{K} := \mathbb{Q}_2$ and $B := \{t \in \mathbb{Q}_2 : |t|_2 \leq 1\}$. Then B^c is a non-metrizable compact additive subgroup of the topological vector space $\mathbb{K}^c$ (here c stands for the cardinality of continuum). The same example shows also that the metrizability of E is essential in Theorem 3(b).

Recall that a *locally* $\mathbb{K}$-*convex* topological vector space E over a locally compact $\mathbb{K}$, i.e. a topological vector space with a basis of absolutely $\mathbb{K}$-convex neighbourhoods of zero, is dually separating, see [33]. If E is not locally $\mathbb{K}$-convex, the above property fails in general. Indeed, the space $E := L^p([0,1], \mathbb{K})$ of all Borel measurable functions on $[0,1]$ with values in a non-archimedean non-trivially valued field $(K, |\ |)$ for which the Lebesgue integral $\|f\|_p := (\int_0^1 |f(x)|^p \, dx)^{1/p}$ is finite $(p \geq 1)$ is not locally $\mathbb{K}$-convex and E does not admit a nontrivial $\mathbb{K}$-convex subset. Therefore $E' = \{0\}$, see [38], p. 131, or [47], p. 89.

Since the weak topology of a metrizable locally convex space over the real or complex numbers is angelic, and has countable tightness see [28, p. 39 and p. 38] (see also [11] and [12] for a large class of spaces enjoying this property), Theorem 2 and Theorem 3 may suggest the following problem (inspired also by a related problem mentioned by Wallace [48, p. 96]: "*What spaces admit what algebraic structures*?").

Problem. Let E be a real locally convex space. For which compact subsets X of E does there exist a compact topological group which is homeomorphic to X?

Clearly every such X must be homogeneous. Also X cannot be convex since the Schauder fixed point theorem fails for compact topological groups (translations do not have a fixed point).

Remark. After sending the paper:

1) R. Buzyakova informed us that the results of Theorem 2 were known. We independently observed that 4) ⇒ 3) of Theorem 2 is covered by [31, Theorem 2.3].

2) A. Arhangels'kii informed us that a statement similar to 3) ⇒ 4) in Theorem 1 also holds in the following context: if X is a topological group which is a p-space in the sense of [2] and $C_p(X)$ is Lindelöf, then X is metrizable and separable.

Acknowledgements. The authors wish to thank Professor D. Dikranjan for his remarks and discussions concerning this article. We also thank the referee for his careful reading of the paper; his comments led us to an improved version of it.

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Received 23 January 2006; revised 16 May 2006; in final form 30 May 2006

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